Bifurcations in Network Dynamical Systems

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CHAPTER 6

SUMMARY

This dissertation explores a new approach to analysing network dynamical systems; that of *hidden symmetry.* The main idea is that a large class of network vector fields can be seen as the restriction of some larger, more symmetrical system. This symmetry may be considered 'hidden' for two reasons: First and foremost, it is often not present in the original network, but only in this larger 'lift'. Nevertheless, it has important consequences for the dynamics of the original network, and can be used to explain some of the more unusual phenomena in network dynamical systems. Secondly, the symmetry in question is not a 'classical' one, whereby a system is preserved under the action of some group. Rather, the linear transformations that respect solutions to the system may not be invertible. Consequently, the symmetry is often not represented by a group, but at most by a semigroup or monoid (i.e. a semigroup with identity).

The main body of this thesis comprises four chapters, each of which is a self-contained scientific article. The first of these introduces the main ideas, by relating them to the more visual language of *graph fibrations.* These may be seen as the natural choice of morphisms between network graphs. The main ingredient here is the presence of a functor between network graphs and network dynamical systems. In particular, the aforementioned lift to a more symmetrical system corresponds to a 'quotient-fibration' between the original network graph and a construct called the *fundamental network.* The symmetries themselves can then be visualised by 'self-fibrations' in the fun-

damental network.

As a symmetry in a dynamical system simply sends solutions to solutions, a great deal of information may be inferred from the existence of such transformations. As it turns out, one can take this a step further; a 'zoom out' to a slightly larger class of network vector fields gives a description of the network structure *in terms of symmetry.* In other words, the networks do not just possess (hidden) symmetry, but represent precisely all those dynamical systems with this property. This means that the rather intractable concept of a network structure is now equivalent to the very workable and well-behaved notion of symmetry. The second article exploits this correspondence by setting up a theory of *center manifold reduction* tailored to the network setting. In particular, it is shown that the resulting *reduced vector fields* may again be interpreted as network vector fields, and that information about *synchrony* (the phenomenon whereby cells behave in unison) is preserved. This allows one to make claims on *generic* events in systems with a given network structure. Most notably, it sheds some light on the mystery of unusual *bifurcations* (sudden, qualitative changes) as a generic phenomenon in many network systems.

The third article builds on the second one by using the inherently algebraic description of symmetry. The main result is that a special structure in a network graph, called a *projection block*, may be omitted from the network to obtain a smaller one. Any generic bifurcation scenario in this new network is also a generic occurrence in the original system, with at most one well-understood exception. This technique is then used to analyse the generic bifurcations in a generalisation of the so-called *feed-forward network.* This should be considered a first step towards a more modular description of network dynamical systems. One whereby a network can be decomposed into smaller components, each giving different information about the original system.

Finally, the fourth article proves a technical result about bifurcations in general, monoid-symmetric systems. It is shown what generalized kernel and center subspace to expect in a generic bifurcation in the presence of a linear monoid-symmetry, and for any number of bifurcation parameters. In particular, it follows that a generic one-parameter steady state bifurcation occurs along one absolutely indecomposable subrepresentation. By the foregoing discussion, these results are directly applicable to the study of bifurcations in network systems. Moreover, they are valid for systems with, for example, a (linear) non-compact Lie-group symmetry. The results are obtained by analysing the set of linear endomorphisms with generalized kernel or center subspace isomorphic to a given subrepresentation.